Mathematics: The Core Course for A-level; L. Bostock and S. Chandler

Chapter 13: Complex Numbers

Exercise 13b

(1) (a) $x^2 + x + 1 = 0$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm 3i}{2}$$

$$= -\frac{1}{2} \pm \frac{3}{2} i$$

(b) $2x^2 + 7x + 1 = 0$

$$\therefore \quad x = \frac{-7 \pm \sqrt{49-8}}{4} = -\frac{7}{4} \pm \frac{41}{4} i$$

(c) $x^2 + 9 = 0$

$$\therefore \quad x^2 = -9 \Rightarrow x = \pm 3i$$

(d) $x^2 + x + 3 = 0$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1-12}}{2} = -\frac{1}{2} \pm \sqrt{11}\, i$$

~~(e) $2x^4$ [illegible] $= 0$~~

(e) $x^4 - 1 = 0$.

This is a quartic & ∴ has 4 Roots. Two Roots can be found as usual:

$$x^4 = 1 \Rightarrow x = \pm 1$$

∴ $x^4 - 1 = 0$ can be factorised as

$$x^4 - 1 = (x-1)(x+1)(ax^2 + bx + c) = 0$$

Now Expand & compare coefficients:

$$x^4 - 1 = (x^2 - 1)(ax^2 + bx + c) = 0$$

$$= ax^4 + bx^3 - ax^2 + cx^2 - bx - c = 0$$

x^4: $a = 1$
x^3: $0 = b$
x^2: $0 = c - a \rightarrow c = a = 1$

So $x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$

∴ $(x^2 - 1) = 0$ OR $x^2 + 1 = 0$

hence $x = \pm 1$ or $x = \pm i$

(2) For i & $-i$ we have

(a)

$$x = i \quad \& \quad x = -i \quad \Rightarrow \quad x - i = 0 \quad \& \quad x + i = 0$$

$$\therefore \quad (x-i)(x+i) = x^2 - i^2 = x^2 + 1 = 0$$

Don't forget to put " $=0$ " since this is needed for i & $-i$ to be called "Roots". In other words, we only get i & $-i$ as answers / Roots when we solve $x^2 + 1 = 0$.

(b) $2+i$ & $2-i$: This implies $x = 2+i$, $x = 2-i$

$$\therefore \quad x - (2+i) = 0 \quad \& \quad x - (2-i) = 0$$

Hence our Equation is

$$[x - (2+i)][x - (2-i)] = 0$$

$$\therefore \quad x^2 - x[(2-i) + (2+i)] + (2+i)(2-i) = 0$$

$$\Rightarrow \quad x^2 - 4x + 5 = 0$$

(c) For $1-3i$ & $1+3i$ we have $x = 1-3i$ & $x = 1+3i$.

$$\therefore \quad [x - (1-3i)][x - (1+3i)] = 0$$

$$\Rightarrow \quad x^2 - x[(1+3i) + (1-3i)] + (1-3i)(1+3i) = 0$$

$$\Rightarrow \quad x^2 - 2x + 10 = 0$$

(d) FOR $1+i$, $1-i$, 2 we have

$x = 1+i$, $x = 1-i$, $x = 2$

$\therefore \; [x-(1+i)][x-(1-i)](x-2) = 0$

$\Rightarrow \; (x^2 - x[(1-i)+(1+i)] + (1+i)(1-i))(x-2) = 0$

$\Rightarrow \; (x^2 - 2x + 2)(x-2) = 0$

$$\therefore \; \begin{aligned} x^3 - 2x^2 & \\ -2x^2 + 4x & \\ +2x - 4 &= 0 \end{aligned}$$

hence $x^3 - 4x^2 + 6x - 4 = 0$

(3) FOR $ax^2 + bx + c = 0$

(a) $x = 2+i \Rightarrow x = 2-i$ is The other Root.

So $-\frac{b}{a} = $ Sum of Roots $= (2+i)+(2-i)$

$= 4$

$\frac{c}{a} = $ PRoduct of Roots $= (2+i)(2-i)$

$= 5$

(b) $x = 3-4i \Rightarrow x = 3+4i$ is The other Root

So $-\frac{b}{a}$ = Sum of Roots $= (3-4i)+(3+4i)$

$= 6$

$\frac{c}{a}$ = Product of Roots $= (3-4i)(3+4i)$

$= 25$

(c) $x = i \Rightarrow x = -i$ is The other Root

So $-\frac{b}{a}$ = Sum of Roots $= i - i = 0$

$\frac{c}{a}$ = Product of Roots $= (i)(-i) = 1$

(d) $x = 5i - 12 \Rightarrow x = -5i - 12$ is the other root

So $-\frac{b}{a}$ = Sum of Roots $= (5i-12)+(-5i-12)$

$= -24$

$\frac{c}{a}$ = Product of Roots $= (5i-12)(-5i-12)$

$= -25i^2 + 144 = 169$

(c) $x = -1 - i \Rightarrow x = -1 + i$ is The other Root

So $-\frac{b}{a}$ = Sum of Roots $= (-1-i)+(-1+i)$

$= -2$

$\frac{c}{a}$ = PRoduct of Roots $= (-1-i)(-1+i)$

$= 2$

(4) (a) $x^2 + 4x + 5$

let $x^2 + 4x + 5 = 0$, Then we can use The formula to find x :
$$x = \frac{-4 \pm \sqrt{16 - 20}}{2}$$
$$= -\frac{4}{2} \pm \frac{\sqrt{-4}}{2} = -2 \pm i$$

So The factoRs are $x - (-2+i)$ & $x - (-2-i)$

i.e $(x+2-i)$ & $(x+2+i)$

(b) $x^2 - 2x + 17$

Let $x^2 - 2x + 17 = 0$ so That we can use The formula to find x :
$$x = \frac{+2 \pm \sqrt{4 - 68}}{2} = 1 \pm 4i$$

hence factoRs are $x - (1+4i)$ & $x - (1-4i)$

i.e $x - 1 - 4i$ & $x - 1 + 4i$

(c) x^2+x+1

Let $x^2+x+1=0$ so that we can use the formula to find x : $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

So factors are

$$x - \left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) \quad \& \quad x - \left(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)$$

i.e. $$x + \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} \quad \& \quad x + \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

(d) x^3-8

Let $x^3-8=0$ so that we can find x

So $x=2 \Rightarrow x-2$ is a factor.

$\therefore\; x^3-8 = (x-2)(ax^2+bx+c)$

$$= ax^3+bx^2-2ax^2-2bx+cx-2c$$

Compare coeffs: x^3 : $a=1$
x^2 : $0 = b-2a \Rightarrow b=2$
x : $0 = -2b+c \Rightarrow c=4$

$\therefore\; x^3-8 = (x-2)(x^2+2x+4)$

For the quadratic : $x = \dfrac{-2 \pm \sqrt{4-16}}{2}$

$$= -1 \pm \sqrt{3}\,i$$

Answer in the Book is wrong

So we have $x - (-1+\sqrt{3}\,i)$ & $x - (-1-\sqrt{3}\,i)$

5 a $$\frac{1}{x^2+1} = \frac{A}{x-i} + \frac{B}{x+i}$$

So $1 = A(x+i) + B(x-i)$

let $x = i$: $1 = 2iA \Rightarrow A = \frac{1}{2i}$

$x = -i$: $1 = -2Bi \Rightarrow B = -\frac{1}{2i}$

So $$\frac{1}{x^2+1} = \frac{1/2i}{x-i} - \frac{1/2i}{x+i}$$

$$= \frac{1}{2i} \cdot \frac{1}{x-i} \cdot \frac{i}{i} - \frac{1}{2i} \frac{1}{x+i} \frac{i}{i}$$

$$= \frac{1}{2} \frac{i}{x+i} - \frac{1}{2} \frac{i}{x-i}$$

b $$\frac{4}{x^2-4x+5} = \frac{4}{[x-(i+2)][x-(-i+2)]}$$

So $$\frac{4}{[x-(i+2)][x-(-i+2)]} = \frac{A}{x-(i+2)} + \frac{B}{x-(-i+2)}$$

$\therefore \quad 4 = A[x-(-i+2)] + B[x-(i+2)]$

let $x = -i+2$: $4 = A(0) - 2iB \Rightarrow B = -\frac{2}{i} = 2i$

$x = i+2$: $4 = A(2i) + B(0) \Rightarrow A = -2i$

So $$\frac{4}{[x-(i-2)][x-(-i-2)]} = \frac{-2i}{x-(i+2)} + \frac{2i}{x-(-i+2)}$$

(c) $$\frac{16}{x^2+4x+8} = \frac{16}{[x+(2i+2)][x-(2i-2)]}$$

Now, $$\frac{16}{[x+(2i+2)][x-(2i-2)]} = \frac{A}{x+(2i+2)} + \frac{B}{x-(2i-2)}$$

$\therefore$

$$16 = A[x-(2i-2)] + B[x+(2i+2)]$$

let $x = 2i-2$: $\quad 16 = A(0) + 4iB \Rightarrow B = \frac{4}{i} = -4i$

$x = -2i-2$: $\quad 16 = -4iA + B(0) \Rightarrow A = -\frac{4}{i} = 4i$

So $$\frac{16}{[x+(2i+2)][x-(2i-2)]} = \frac{4i}{x+(2i+2)} - \frac{4i}{x-(2i-2)}$$

(d) $$\frac{2}{x^2+4} = \frac{2}{(x-2i)(x+2i)}$$

Now, $$\frac{2}{(x-2i)(x+2i)} = \frac{A}{x-2i} + \frac{B}{x+2i}$$

hence $$2 = A(x+2i) + B(x-2i)$$

let $x = 2i$: $\quad 2 = 4iA \Rightarrow A = \frac{1}{2}\frac{1}{i} = -\frac{i}{2}$

$x = -2i$: $\quad 2 = -4iB \Rightarrow B = -\frac{1}{2}\frac{1}{i} = \frac{i}{2}$

So $$\frac{2}{x^2+4} = -\frac{i}{2(x-2i)} + \frac{i}{2(x+2i)}$$

(e) $$\frac{x+8}{x^2+4x+13} = \frac{x+8}{[x-(-3i-2)][x-(3i-2)]}$$

$$\frac{x+8}{[x-(-3i-2)][x-(3i-2)]} = \frac{A}{x-(-3i-2)} + \frac{B}{x-(3i-2)}$$

So $$x+8 = A[x-(3i-2)] + B[x-(-3i-2)]$$

Let $x = 3i-2$: $$3i-2+8 = A(0) + 6iB$$

$$\therefore B = \frac{3i+6}{6i} = \frac{3i+6}{6i}\cdot\frac{i}{i}$$

$$= \frac{1}{2} - i$$

Let $x = -3i-2$: $$-3i-2+8 = A[-3i-2-3i+2] + B(0)$$

$$-3i+6 = -6iA$$

$$\therefore A = \frac{3i-6}{6i} = \frac{3i-6}{6i}\cdot\frac{i}{i}$$

$$= \frac{1}{2} + i$$

So $$\frac{x^2+8}{x^2+4x+13} = \frac{\frac{1}{2}+i}{x-(-3i-2)} + \frac{\frac{1}{2}-i}{x-(3i-2)}$$

(6) $1+\omega+\omega^2=0$ (*)

(a) $$\frac{(1+\omega)^2}{\omega} = \frac{1+\omega+\omega^2}{\omega} = 0$$

(b) $(1+2\omega+3\omega^2)(3+2\omega+\omega^2)$

$$= [(1+\omega+\omega^2)+(\omega+2\omega^2)][(1+\omega+\omega^2)+(2+\omega)]$$

The moral here is not to expand in the usual way, but to group terms according to (*). This makes things quicker:

$$\therefore (1+2\omega+3\omega^2)(3+2\omega+\omega^2) = [0+(\omega+\omega^2)+\omega^2]$$
$$\times[0+(1+\omega)+1]$$
$$= [-1+\omega^2][1-\omega^2]$$
$$= -1+2\omega^2-\omega^4$$
$$= -1+2\omega^2-\omega.\omega^3$$

Since $\omega^3=1$ we have $(1+2\omega+3\omega^2)(3+2\omega+\omega^2)$

$$= -1+2\omega^2-\omega = -1-\omega+2\omega^2$$
$$= \omega^2+2\omega^2 \text{ by (*)}$$
$$= 3\omega^2$$

(c) $\omega^7+\omega^8+\omega^9 = \omega^3(\omega^4+\omega^5+\omega^6)$

$= \omega^4+\omega^5+\omega^6$, since $\omega^3=1$

$= \omega^3(\omega+\omega^2+\omega^3)$

$= \omega+\omega^2+\omega^3$, since $\omega^3=1$

$= \omega(1+\omega+\omega^2) = 0$ (!) (by (*))

(7) $x^3+1=0 \Rightarrow x^3=-1 \Rightarrow x=-1$ is one root.

$\therefore (x+1)(ax^2+bx+c) = x^3+1$

$\therefore \quad ax^3+bx^2+ax^2+cx+bx+c = x^3+1$

compare coeffs: $\underline{\underline{x^3}}$: $a=1$

$\underline{\underline{x^2}}$: $b+a=0 \Rightarrow b=-1$

$\underline{\underline{x}}$: $c+b=0 \Rightarrow c=1$

So $x^3+1 = (x+1)(x^2-x+1)=0$

For the quadratic we get $x = \dfrac{+1 \pm \sqrt{1-4}}{2}$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

So the three roots of -1 are $-1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$

Let $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

Then $\lambda^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

So $\lambda - \lambda^2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= 1$

So $-1 + \lambda - \lambda^2 = 0. \implies \lambda = 1 + \lambda^2$